Journal of Statistical Physics, Vol. 109, Nos. 3/4, November 2002 (© 2002)

Long-Time Tails, Weak Localization, and Classical and Quantum Critical Behavior

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Received January 8, 2002; accepted March 5, 2002

An overview is given of the long-time and long-distance behavior of correlation functions in both classical and quantum statistical mechanics. After a simple derivation of the classical long-time tails in equilibrium time correlation functions, we discuss analogous long-distance phenomena in nonequilibrium classical systems. The paper then draws analogies between these phenomena and similar effects in quantum statistical mechanics, with emphasis on the soft modes that underly long-time tails and related phenomena. We also elucidate the interplay between critical phenomena and long-time tails, using the classical liquid-gas critical point and the quantum ferromagnetic transition as examples.

KEY WORDS: Long-time tails; generic scale invariance; weak localization; critical behavior; quantum phase transitions

1. INTRODUCTION

It has been known for some time that equilibrium time correlation functions generically show a power-law temporal decay in the limit of long times. If this behavior is not obviously related to any conservation law, or to soft modes in the system, long-time tails (LTTs) is the term traditionally applied to this phenomenon. Boltzmann-type kinetic equations usually

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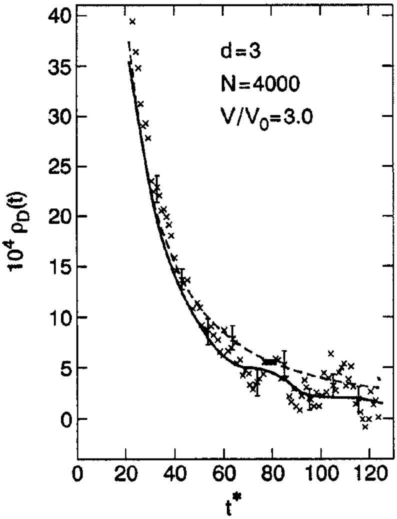

Fig. 1. Normalized velocity autocorrelation function $\rho_D(t) = \langle \mathbf{v}(t) \cdot \mathbf{v}(0) \rangle / \langle \mathbf{v}^2(0) \rangle$ as a function of the dimensionless time $t^* = t/t_0$, where t_0 is the mean-free time. The crosses indicate computer results obtained by Wood and Erpenbeck, see ref. 4, for a system of 4000 hard spheres at a reduced density corresponding to $V/V_0 = 3$, where V is the actual volume and V_0 is the close-packing volume. The dashed curve represents the theoretical curve $\rho_D(t) = \alpha_D\,(t^*)^{-3/2}$. The solid curve represents a more complete evaluation of the mode-coupling formula with contributions from all possible hydrodynamic modes and with finite-size corrections included, see ref. 5. From ref. 6.

predict an exponential decay in time, which is why the discovery of LTTs came as a considerable surprise. An example is the velocity auto-correlation function in a hard-sphere fluid, which is shown in Fig. 1. In accord with the Boltzmann–Enskog equation, initially the decay is exponential, with a time scale given by the mean-free time t_0 between particle collisions. However, for longer times, $t \gg t_0$, it decays algebraically, as $t^{-3/2}$, and more generally as $t^{-d/2}$ in d-dimensions. This striking result was first observed in computer simulations by Alder and Wainwright,(1) and then understood theoretically by Dorfman and Cohen,(2) and by Ernst, Hauge, and van Leeuwen.(3) The velocity autocorrelation function is of particular importance because it determines the frequency-dependent self-diffusion coefficient $D(\omega)$ via

$$D(\omega) = \frac{1}{d} \int_0^\infty dt\, e^{i\omega t} \langle \mathbf{v}(t) \cdot \mathbf{v}(0) \rangle. \tag{1}$$

Here $\mathbf{v}(t)$ is the velocity of a tagged particle at time t, and the angular brackets denote an equilibrium ensemble average. The slow algebraic decay of the velocity autocorrelation function implies a low-frequency behavior

$$D(\omega \to 0)/D(0) = 1 - c_d^D\,(i\omega t_0)^{(d-2)/2} + \cdots \qquad (2 < d < 4), \tag{2}$$

$$D(\omega \to 0)/D(0) = 1 - c_2^D\,\ln(i\omega t_0) + \cdots \qquad (d = 2), \tag{3}$$

with $c_d^D > 0$. This result indicates that in $d \leqslant 2$ the zero-frequency diffusion coefficient does not exist, which in turn implies that the ordinary local hydrodynamic description of a fluid is not valid in low dimensions. This conclusion is indeed correct, as detailed investigations show.$^{(7)}$ Note that in this classical case transport in low dimensions is faster than diffusive. More generally, it is known that all the transport coefficients in a fluid are given by time integrals over appropriate time correlation functions, and that all these time correlation functions have similar LTTs.$^{(8)}$

In contrast to this behavior of time correlation functions, the spatial decay of static correlations at large distances in classical systems in equilibrium is generically indeed exponential. An exception is the behavior at critical points. It was known long before the discovery of LTTs that at these isolated points in the phase diagram correlations decay algebraically in both time and space, i.e., like power-laws with universal critical exponents.$^{(9-11)}$ Algebraic decay implies scale invariance, i.e., the correlation functions are generalized homogeneous functions of space and time. Correlations that exhibit scale invariance in an entire region of the phase diagram, rather than just at isolated points, are said to exhibit generic scale invariance (GSI).$^{(6,\,12)}$ In this language, the LTTs are an example of GSI in the time domain.

A natural question is what happens to the correlation functions that exhibit GSI as a critical point is approached, and how these correlations influence the critical behavior itself. We will see below that the mechanism that causes the LTTs in the entire fluid phase typically becomes amplified near critical points and leads to nontrivial (i.e., non-van Hove) critical dynamics, or critical singularities in various transport coefficients. Violation of the van Hove picture for the slowing down of critical fluctuations became apparent when the thermal conductivity of fluids was found to diverge at the critical point, while the viscosity of liquid mixtures exhibited a weak divergence at the consolute point.$^{(13)}$ These singularities were later understood to be due to mode-coupling effects by Fixman,$^{(14)}$ and then by Kadanoff and Swift$^{(15)}$ and by Kawasaki.$^{(16)}$ Here mode-coupling means the coupling of products of slow hydrodynamic modes, or excitations, that are in some sense orthogonal to single hydrodynamics modes, to the single modes whose correlations form the correlation functions in question. Historically, the LTTs were discovered after these critical singularities were observed, and it turned out that they, too, are caused by mode-coupling effects.(3)

In a classical system in a nonequilibrium steady state (NESS) the situation is still more complicated. As we will discuss at the end of the first part of the present paper, in this case the same mode-coupling correlations that lead to the equilibrium LTTs and to the critical singularities, also lead to GSI in spatial correlations.

In recent years there has been a separate but analogous development in the study of electronic systems at zero temperature.(17) In this case the mode-coupling effects are stronger than in classical systems, leading to even more dramatic effects. For example, they lead to the localization of electrons in disordered two-dimensional systems, even for arbitrarily weak disorder.(18) In addition, because of the coupling of the electron dynamics to the various electronic correlation functions, and because of the coupling between statics and dynamics that is inherent in quantum statistical mechanics,(19) it turns out that in zero-temperature systems, GSI exists in both spatial and temporal correlations even in equilibrium. In disordered systems these many-body quantum phenomena are known as weak-localization effects,(20) since they can be considered precursors to an Anderson or Anderson–Mott metal-insulator transition,(21–23) but analogous, if slightly weaker, effects also occur in clean quantum systems. In the second part of this paper we explain these phenomena, making connections with the GSI in classical systems. We then explore how these generic correlations influence, and are themselves modified by, critical behavior at quantum phase transitions. The latter are defined as phase transitions that occur at zero temperature and are triggered by some non-thermal control parameter, like pressure or composition.(24–26) The metal-insulator transitions mentioned above are examples of quantum phase transitions.

As noted above, in the classical case the basic source of the power-law decay characteristic of GSI is two-fold. First, soft or massless modes must exist. Their softness can arise either from some basic underlying conservation law, or they can be Goldstone modes, i.e., arise from a spontaneously broken continuous symmetry. Second, the theory must in some sense be nonlinear, so that these soft modes couple to the physical observables. It is this coupling that is missing in simple theories, and this omission results in the erroneous prediction of exponential decay. In both classical and quantum systems such coupling mechanisms exist, and in this way the GSI effects found in both systems are related.

The above discussion illustrates that it is incorrect to view LTT effects, as it is sometimes done, as only leading to small corrections to transport coefficients, with no deeper importance. To the contrary, as we have pointed out, they are responsible for a myriad of effects. Not only do LTTs cause the leading frequency and, in the quantum case, temperature corrections to the transport coefficients, but there are numerous cases where their effects are greatly magnified, and they dominate the physics both qualitatively and quantitatively. Examples include, systems in reduced dimensionalities, systems near critical points, systems in Goldstone phases, and nonequilibrium systems. One of the goals of this paper is to illustrate how these effects come about, and how they are connected.

We will proceed as follows. In Section 2 we present the dynamical equations that describe fluctuations in a classical fluid. We use these equations to first describe the LTTs in a classical fluid, and then to describe critical dynamics near a liquid-gas critical point. We conclude this section by describing the GSI that occurs in spatial correlations in a nonequilibrium fluid. In Section 3 we consider the weak-localization effects in disordered electron systems, which are quantum analogs of the classical LTTs. Because in quantum statistical mechanics these dynamical correlations couple to the static correlations, they lead to GSI in spatial correlations even in equilibrium systems, as long as they are at zero temperature. We then discuss how this GSI is important for describing quantum phase transitions such as the ferromagnetic transition in itinerant electron systems at zero temperature. This amounts to a discussion of how the quantum LTTs affect quantum phase transitions. We also discuss how nonequilibrium effects lead to even longer-range correlations in quantum systems at zero temperature. We conclude with a discussion of our results in Section 4.

2. GSI AND PHASE TRANSITIONS IN CLASSICAL SYSTEMS

2.1. Fluctuating Hydrodynamics

To be specific, let us consider a classical fluid as an example of a finite temperature, or classical, statistical mechanics system. In general we are interested in the long-time and large-distance behavior of systems, so we will always be concerned with the identification of the relevant soft, or hydrodynamic, variables that determine this behavior. In our case they are the conserved variables of mass density ρ, momentum density $\mathbf{g} = \rho\mathbf{u}$, with $\mathbf{u}$ the fluid velocity, and energy density ϵ, and the relevant low-frequency and long-wavelength dynamical equations are the Navier–Stokes equations. To describe fluctuations, a Langevin force is added so that the equations are(27)

$$\partial_t \rho + \nabla \cdot \mathbf{g} = 0, \tag{4}$$

$$\partial_t g_\alpha + \nabla_\beta (g_\alpha u_\beta) = -\nabla_\alpha p + \nabla_\beta \Bigg[\eta(\nabla_\alpha u_\beta + \nabla_\beta u_\alpha) + \left(\zeta - \frac{(d-1)}{d} \eta \right) \delta_{\alpha\beta} (\nabla \cdot \mathbf{u}) + P_{\alpha\beta} \Bigg], \tag{5}$$

$$\rho T (\partial_t + \mathbf{u} \cdot \nabla) s = \nabla_\alpha (\lambda \nabla_\alpha T + q_\alpha). \tag{6}$$

Here p is the pressure, T the temperature, η the shear viscosity, ζ the bulk viscosity, and λ the thermal conductivity, and a summation over repeated indices is implied. We have chosen to use the entropy density s, rather than the energy ϵ, as our hydrodynamic variable, and in Eq. (6) we have neglected a viscous dissipation term that represents entropy production since it is irrelevant to both the leading LTTs, and to the leading critical dynamics singularities. The Langevin forces $P_{\alpha\beta}$ and q_α are uncorrelated with the initial velocity, and satisfy

$$\langle P_{\alpha\beta}(\mathbf{x}, t)\, P_{\mu\nu}(\mathbf{x}', t') \rangle = 2 k_{\mathrm{B}} T \Bigg[\eta(\delta_{\alpha\mu}\delta_{\beta\nu} + \delta_{\alpha\nu}\delta_{\beta\mu}) + \left(\zeta - \frac{(d-1)}{d} \eta \right) \delta_{\alpha\beta}\delta_{\mu\nu} \Bigg] \delta(\mathbf{x}-\mathbf{x}')\, \delta(t-t'), \tag{7}$$

$$\langle q_\alpha(\mathbf{x}, t)\, q_\beta(\mathbf{x}', t') \rangle = 2 k_{\mathrm{B}}\, \lambda\, T^2\, \delta_{\alpha\beta}\, \delta(\mathbf{x}-\mathbf{x}')\, \delta(t-t'), \tag{8}$$

$$\langle P_{\alpha\beta}(\mathbf{x}, t)\, q_\mu(\mathbf{x}', t') \rangle = 0, \tag{9}$$

where k_{B} is Boltzmann's constant. The above equations can be derived in a number of ways and are known to exactly describe the long-wavelength and low-frequency fluctuations in a fluid.(27)

2.2. Long-Time Tails

To illustrate the LTTs we choose a slightly different example than the self-diffusion coefficient discussed in the Introduction, namely, the shear viscosity of a classical fluid. We present a simplified calculation of this transport coefficient, and then discuss more general results. For simplicity, we assume an incompressible fluid. The mass conservation law is then trivial,

$$\nabla \cdot \mathbf{u}(\mathbf{x}, t) = 0, \tag{10}$$

and the momentum-conservation law is a closed PDE for $\mathbf{u}$,

$$\partial_t u_\alpha + (\mathbf{u} \cdot \nabla) u_\alpha = -\nabla_\alpha (p/\rho) + \nu \nabla^2 u_\alpha + \nabla_\beta P_{\alpha\beta}/\rho. \tag{11}$$

Here $\nu = \eta/\rho$ is the kinematic viscosity, which we assume to be constant. The only role of the pressure term is to ensure that the incompressibility condition is satisfied. In fact, it can be eliminated by taking the curl of Eq. (11), which turns it into an equation for the transverse velocity, see ref. 28 and below. The cause of the LTTs is the coupling of slow hydrodynamic modes due the nonlinear term in Eq. (11). We treat that term as a perturbation, i.e., we formally multiply it by a coupling constant γ (whose physical value is unity), and calculate its effect to first order in γ.

We first define the velocity autocorrelation function,

$$C_{\alpha\beta}(\mathbf{k}, t) = \langle u_\alpha(\mathbf{k}, t)\, u_\beta(-\mathbf{k}, 0)\rangle, \tag{12}$$

with $\mathbf{k}$ the spatial Fourier transform variable. An equation for C can be obtained by Fourier transforming Eq. (11), multiplying by $u_\beta(-\mathbf{k}, 0)$, and averaging over the noise while keeping in mind that the noise is uncorrelated with the initial fluid velocity. In the case of an incompressible fluid we need to consider only the transverse-velocity correlation function, $C_\perp$. This is easily done by multiplying with unit vectors, $\hat{k}_\perp^{(i)}$ $(i = 1,..., d-1)$, that are perpendicular to $\mathbf{k}$, which eliminates the pressure term. We obtain

$$(\partial_t + \nu \mathbf{k}^2)\, C_\perp(\mathbf{k}, t) = -i\gamma k_\mu \hat{k}^{(i)}_{\perp\alpha} \hat{k}^{(i)}_{\perp\beta} \sum_{\mathbf{q}} \langle u_\mu(\mathbf{k}-\mathbf{q}, t)\, u_\alpha(\mathbf{q}, t)\, u_\beta(-\mathbf{k}, 0)\rangle, \tag{13}$$

where we have used the incompressibility condition, Eq. (10), to write all gradients as external ones. To zeroth order in γ we find, with the help of the f-sum rule(29, 30) $C_\perp(\mathbf{k}, t=0) = k_B T/\rho$,

$$C_\perp(\mathbf{k}, t) = (k_B T/\rho)\, e^{-\nu \mathbf{k}^2 t} + O(\gamma). \tag{14}$$

This is the standard result obtained from linearized hydrodynamics, which predicts exponential decay for $\mathbf{k} \neq 0$. To calculate corrections due to the nonlinearity, we need an equation for the three-point correlation function in Eq. (13). The simplest way to do this is to use the time translational invariance property of this correlation function to put the time dependence in the last velocity, and then use Eq. (11) again.(30) The result is an equation for the three-point function in terms of a four-point one that is analogous to Eq. (14). To solve this equation, we note that, due to the velocity being odd under time reversal, the equal-time three-point correlation function vanishes. By means of a Laplace transform, one can therefore express the

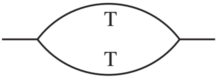

Fig. 2. Leading contribution to the self energy Σ. The internal propagators are given by the transverse velocity autocorrelation function.

three-point function as a product (in frequency space) of the zeroth order result for $C_\perp$, Eq. (14), and the four-point function. To leading (i.e., zeroth) order in γ the latter factorizes into products of velocity autocorrelation functions. Upon transforming back into time space, and to quadratic order in the coupling constant γ, we thus obtain$^{(31)}$

$$(\partial_t + \nu \mathbf{k}^2)\, C_\perp(\mathbf{k}, t) + \int_0^t d\tau \Sigma(\mathbf{k}, t-\tau)\, C_\perp(\mathbf{k}, \tau) = 0, \tag{15}$$

with

$$\Sigma(\mathbf{k}, t) = \gamma^2 \frac{\rho}{k_\mathrm{B} T} k_\mu k_\nu \hat{k}^{(i)}_{\perp,\alpha} \hat{k}^{(i)}_{\perp,\beta} \sum_{\mathbf{p}} [C_{\alpha\beta}(\mathbf{p}, t) \\ \times C_{\mu\nu}(\mathbf{k}-\mathbf{p}, t) + C_{\alpha\nu}(\mathbf{p}, t)\, C_{\mu\beta}(\mathbf{k}-\mathbf{p}, t)] + O(\gamma^3). \tag{16}$$

The self-energy Σ is proportional to $\mathbf{k}^2$, and thus provides a renormalization of the bare viscosity ν. For later reference we mention that diagrammatically the above contribution to Σ can be represented by the one-loop diagram shown in Fig. 2. Equation (16), and its representation by Fig. 2, illustrate the meaning of the term "mode-coupling" as explained in the Introduction.

The source of the LTTs is now evident. In our model of an incompressible fluid, only the transverse component of $C_{\alpha\beta}$ is nonzero,

$$C_{\alpha\beta}(\mathbf{p}, t) = (\delta_{\alpha\beta} - \hat{p}_\alpha \hat{p}_\beta)\, C_\perp(\mathbf{p}, t). \tag{17}$$

Putting $\gamma = 1$, and defining $\delta\nu(t) = \lim_{\mathbf{k}\to 0} \Sigma(\mathbf{k}, t)/\mathbf{k}^2$, one obtains for asymptotically long times

$$\delta\nu(t) = \frac{k_B T}{\rho} \left(\frac{d^2 - 2}{d(d+2)} \right) \frac{1}{(8\pi\nu t)^{d/2}}. \tag{18}$$

This is the well-known contribution of the transverse-velocity modes to the LLT of the viscosity.$^{(32)}$ In a compressible fluid, a similar process coupling two longitudinal modes also contributes to the leading LTT. The other transport coefficients, e.g., λ and ζ, also show LTTs proportional to $1/t^{d/2}$, and all of them have also less leading LTTs proportional to $1/t^{(d+1)/2}$ or weaker.(5, 32)

The correction to the kinematic viscosity is given by the time integral over $\delta\nu(t)$, and for the frequency-dependent kinematic viscosity the Fourier transform of $\delta\nu(t)$ implies a nonanalyticity at zero frequency. More generally, for the frequency or wavenumber-dependent transport coefficients, the LTTs imply the asymptotic forms

$$\mu(\omega)/\mu(0) = 1 - c_d^\mu\, \omega^{(d-2)/2} + \cdots, \tag{19}$$

$$\mu(\mathbf{k})/\mu(0) = 1 - b_d^\mu\, |\mathbf{k}|^{d-2} + \cdots, \tag{20}$$

where μ represents any of the transport coefficients, ν, ζ, λ, or D, in a fluid. The prefactors c_d^μ and b_d^μ are positive.

Of particular interest for the critical dynamics near the liquid-gas critical point is a contribution to $\delta\lambda$ from the coupling of the transverse-velocity fluctuations to the entropy fluctuations. Before performing the wavenumber integral, this particular contribution, which we denote by $\delta\lambda_{\perp s}$, is(33, −11)

$$\delta\lambda_{\perp s}(\mathbf{k}, t) = \frac{1}{\rho^2} \sum_{\mathbf{p}} \chi(\mathbf{p}) \sum_i (\hat{\mathbf{k}} \cdot \hat{\mathbf{p}}_\perp^{(i)})^2\, e^{-(\nu \mathbf{p}_-^2 + D_T \mathbf{p}_+^2)\, t}. \tag{21}$$

Here $D_T = \lambda/\rho\, c_p$ is the thermal diffusivity in terms of c_p, the specific heat at constant pressure. χ is the order parameter susceptibility for the phase transition, where we have anticipated that near the critical point we will need the momentum-dependent χ, and $\mathbf{p}_\pm = \mathbf{p} \pm \mathbf{k}/2$. Setting $\mathbf{k} = 0$ and carrying out the momentum integral leads to a $t^{-d/2}$ LTT. The correction to the thermal conductivity is obtained by integrating $\delta\lambda(t)$ over all times, as in the case of the kinematic viscosity.

These results imply that for $d \leqslant 2$, conventional hydrodynamics does not exist. Indeed, it is now known that for these dimensions the hydrodynamic equations are nonlocal in space and time. For a discussion of this topic we refer the reader elsewhere.(7)

By examining Eq. (16) or (21) one easily identifies a mechanism by which the LTT effects can become even stronger. Consider a system with long-range static correlations, for instance due to Goldstone modes, or due to the vicinity of a continuous phase transition. In either case, some susceptibilities, e.g., the χ in Eq. (21), become long-ranged, amplifying the LTT effect. In the next section we discuss the realization of this scenario in the vicinity of a phase transition. We note that in certain liquid crystal phases some susceptibilities behave as $1/\mathbf{k}^2$, amplifying the LTT effect and causing a breakdown of local hydrodynamics for all dimensions $d \leqslant 4$.(34)

We make one last point concerning the LTTs. So far we have stressed the leading tails that decay like $t^{-d/2}$, but their are numerous subleading LTTs as well. Most of them are uninteresting, but one becomes important near the critical point, via the mechanism discussed in the last paragraph. According to Eq. (21), a central quantity for determining the critical contribution to λ is the shear viscosity η (which enters ν). It turns out that the contribution to η that is dominant near the critical point is a subleading LTT away from criticality. It involves a coupling of two heat or entropy modes and is given by[(33)]

$$\delta\eta(\mathbf{k},t)=\frac{A}{\mathbf{k}^2}\sum_{\mathbf{p}}\chi(\mathbf{p})\,\chi(\mathbf{k}-\mathbf{p})\left(\frac{1}{\chi(\mathbf{p})}-\frac{1}{\chi(\mathbf{k}-\mathbf{p})}\right)^2 \times(\hat{\mathbf{k}}_\perp\cdot\mathbf{p})^2\,e^{-D_T[\mathbf{p}^2+(\mathbf{k}-\mathbf{p})^2)]\,t}, \tag{22}$$

with A a constant. Away from the critical point, this LTT decays as $t^{-(d/2+2)}$, so in fact it is a next-to-next leading LTT. Nevertheless, it is the dominant mode-coupling contribution to η near the critical point because of the two factors of χ in the numerator of Eq. (22). We discuss this enhancement of LTTs by critical susceptibilities further in the next subsection.

2.3. Critical Dynamics

Near continuous phase transitions, fluctuations grow and ultimately diverge at the critical point. For the liquid-gas critical point the order parameter is the difference between the density and the critical density, and the divergent fluctuations are the density fluctuations as described by the density susceptibility. The susceptibility χ in Eq. (21) is proportional to this divergent susceptibility, which in the Ornstein–Zernike approximation behaves as

$$\chi(\mathbf{p})\propto\frac{1}{\mathbf{p}^2+|r|}, \tag{23}$$

with $\mathbf{p}$ measured in suitable units, and r the dimensionless distance from the critical point. Carrying out the time integral, the leading singular contribution to the static, wavenumber-dependent thermal conductivity is[(11)]

$$\delta\lambda(\mathbf{k})=\frac{1}{\rho^2}\sum_{\mathbf{p}}\chi(\mathbf{p})\,\frac{\sum_i(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}}_\perp^{(i)})^2}{\nu\mathbf{p}_-^2+D_T\mathbf{p}_+^2}. \tag{24}$$

Using Eq. (23) in this equation we see that the homogeneous thermal conductivity is infinite at the critical point for all $d\leqslant 4$, diverging as $|r|^{-(4-d)/2}$. This is result of the amplification of the LTT by the critical fluctuations.

By the same mechanism, we see that Eq. (23) leads to a logarithmically singular contribution to $\delta\eta$, if we take into account that Eq. (24) implies that at the critical point, $D_T(\mathbf{k}) \sim |\mathbf{k}|^{d-2}$.

The above are one-loop calculations that use the Ornstein–Zernike susceptibility. To go beyond these approximations one needs to (1) use the correct scaling form for the susceptibility, and (2) use either a self-consistent mode-coupling theory,(15) or a renormalization group approach(35, 11) to improve on the one-loop approximation. The result(11) is that the thermal conductivity diverges like

$$\lambda \propto |r|^{-0.57}, \tag{25}$$

and that the thermal diffusivity vanishes like

$$D_T \propto |r|^{0.67}. \tag{26}$$

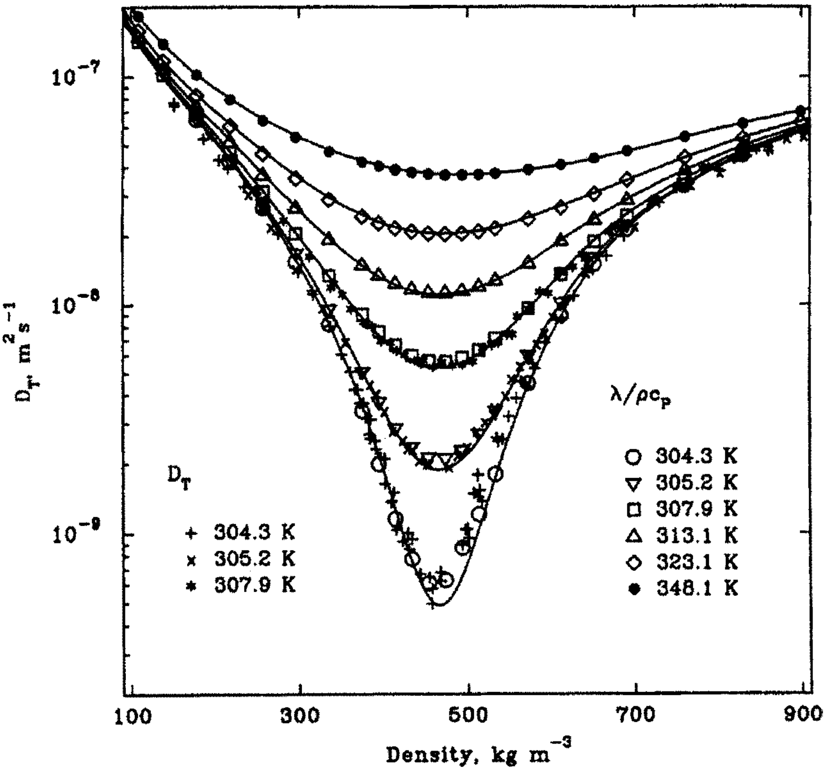

Fig. 3. The thermal diffusivity $D_T = \lambda/\rho\, c_p$ of carbon dioxide in the critical region as a function of density at various temeratures ($T_c = 304.12$ K). The symbols indicate experimental data for D_T measured directly, and for $\lambda/\rho\, c_p$ deduced from thermal-conductivity data. The solid curves represent values calculated from the mode-coupling theory. From ref. 36.

This is in good agreement with experimental results, as shown in Fig. 3, which compares experimental and theoretical results for the thermal diffusivity of carbon dioxide in the critical region.

2.4. Nonequilibrium Effects

We now turn to a fluid in a nonequilibrium steady state. It has been known for some time that these systems in general exhibit GSI in spatial correlations, and that the order parameters for phase transitions in such systems couple to these long-range correlations. The first study of a phase transition of this type considered phase separation in a binary liquid under shear.(37) The spatial correlations responsible for the GSI are closely related to those that lead to the LTTs in equilibrium time correlation functions. We will consider a fluid in a steady, spatially uniform temperature gradient ∇T, but far away from any convective instability. Further, we use a number of approximations that enable us to focus on the most interesting effects of such a temperature gradient. For a justification of this procedure, as well as the underlying details, we refer to ref. 38.

Focusing on the coupling between fluctuations of the transverse fluid velocity $\mathbf{u}_\perp$ and temperature fluctuations δT, and neglecting the nonlinearity in Eq. (5), Eqs. (2.1) can be written

$$\partial_t u_{\perp,\alpha} = \nu \nabla^2 u_{\perp,\alpha} + \frac{1}{\rho_0} (\nabla_\beta P_{\alpha\beta})_\perp, \tag{27}$$

$$\partial_t\, \delta T + (\mathbf{u}\cdot\nabla)\, T = D_T \nabla^2 \delta T + \frac{1}{\rho_0 T_0 C_p} (\nabla\cdot\mathbf{q}), \tag{28}$$

where ρ_0 and T_0 indicate average values. These bilinear equations can be solved by means of Fourier and Laplace transformations. Focusing on static or equal-time correlations, one finds, for example,(38)

$$\langle |\delta\rho(\mathbf{k})|^2 \rangle = \rho k_{\mathrm{B}} T \left(\frac{\partial\rho}{\partial p}\right)_T + \frac{\rho k_{\mathrm{B}} T (\alpha_T \hat{\mathbf{k}}_\perp \cdot \nabla T)^2}{D_T(\nu + D_T)\, \mathbf{k}^4}, \tag{29}$$

$$\langle (\hat{\mathbf{k}}_\perp \cdot \mathbf{g}(\mathbf{k}))\, \delta\rho(-\mathbf{k}) \rangle = \rho k_{\mathrm{B}} T \alpha_T \frac{\hat{\mathbf{k}}_\perp \cdot \nabla T}{(\nu + D_T)\, \mathbf{k}^2}, \tag{30}$$

with,

$$\alpha_T = \frac{-1}{\rho} \left(\frac{\partial\rho}{\partial T}\right)_p, \tag{31}$$

the thermal expansion coefficient.

There are several remarkable aspects of these results. First, Eq. (29) for the density correlations implies that the first term, which also exists in equilibrium, is delta-function correlated in real space, while the second term decays like const. $-|\mathbf{x}|$ in three-dimensions, with $|\mathbf{x}|$ the distance in real space.(39, 40) Equation (30) shows that the transverse momentum-density correlation functions decays as $1/|\mathbf{x}|$ in three-dimensions. Both of these results show that spatial correlations in a NESS exhibit GSI. Second, the right-hand side of Eq. (30) is essentially the integrand in Eq. (24), which shows the close connection between the LTTs, singularities in transport coefficients near continuous phase transitions, and the GSI of spatial correlations in a NESS.

The nonequilibrium fluctuations arise from a coupling between the temperature fluctuations perpendicular to the temperature gradient and the transverse-momentum (viscous) fluctuations parallel to the temperature gradient.(38, 39, 41) The amplitudes of these fluctuations can be measured by

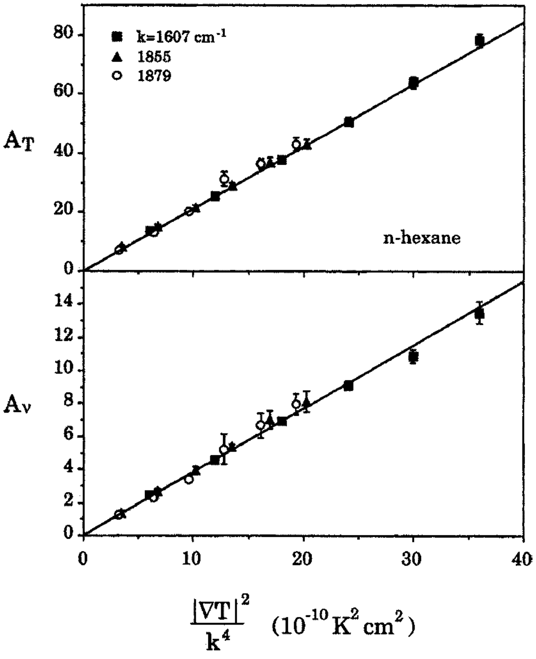

Fig. 4. Amplitudes A_T and A_v of the nonequilibrium temperature and transverse-momentum (viscous) fluctuations in liquid hexane at 25°C as a function of $(\nabla T)^2/\mathbf{k}^4$. The symbols indicate experimental data. The solid lines represent the values predicted by Eqs. (31). From ref. 42.

small-angle light-scattering experiments. Specifically, the dynamical structure factor $S_{\rho\rho}(\mathbf{k}, t) = \langle \delta\rho(\mathbf{k}, t)\, \rho(-\mathbf{k}, 0)\rangle$, which is proportional to the scattering cross section, in a NESS has the form[(6)]

$$S_{\rho\rho}(\mathbf{k}, t) = S_0[(1 + A_T)\, e^{-D_T \mathbf{k}^2 t} - A_\nu\, e^{-\nu \mathbf{k}^2 t}], \tag{32}$$

where S_0 is the structure factor in equilibrium, and

$$A_T = \frac{c_p \nu / D_T}{T(\nu^2 - D_T^2)} \frac{(\hat{\mathbf{k}}_\perp \cdot \nabla T)^2}{\mathbf{k}^4}, \tag{33}$$

$$A_\nu = \frac{c_p}{T(\nu^2 - D_T^2)} \frac{(\hat{\mathbf{k}}_\perp \cdot \nabla T)^2}{\mathbf{k}^4}. \tag{34}$$

For $t = 0$ one recovers the equal-time density correlation function, Eq. (29). The amplitudes A_T and A_ν are proportional to $(\nabla T)^2/\mathbf{k}^4$, which has been verified by experiments, see Fig. 4. The agreement, with no adjustable parameters, is excellent. Notice that the amplitude of the temperature fluctuations is enhanced by a factor of a hundred compared to the scattering by an equilibrium fluid.

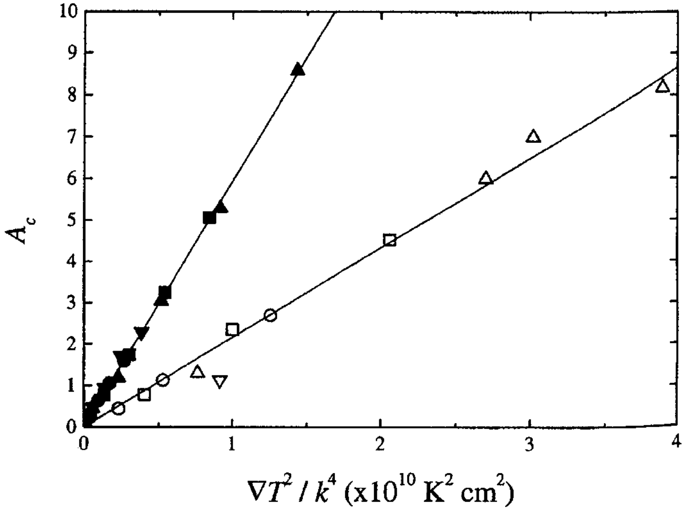

Fig. 5. Amplitude A_c of nonequilibrium concentration fluctuations in polystyrene and toluene solutions at 25°C as a function of $(\nabla T)^2/\mathbf{k}^4$. The symbols indicate experimental data at polymer mass fractions $w = 4.00\%$ (solid symbols) and $w = 0.50\%$ (open symbols), respectively. Different symbol shapes correspond to different values of the wave number k. The solid lines are linear fits to the experimental data in agreement with the theoretical predictions.[(45)]

In a liquid mixture or a polymer solution, a temperature gradient induces a concentration gradient due to the Soret effect. Nonequilibrium concentration fluctuations then arise from a coupling between the concentration fluctuations and the transverse-momentum fluctuations.(43, 44) Just as in the case of the nonequilibrium temperature and viscous fluctuations, the amplitude of the nonequilibrium fluctuations will also be proportional to $(\nabla T)^2/\mathbf{k}^4$. The presence of such long-range nonequilibrium concentration fluctuations has also been confirmed experimentally, see Fig. 5.

One final point concerns the question what these generic long-range spatial correlations would do to critical behavior. This is a natural question, as usually long-range interactions, or long-range correlations, fundamentally modify important aspects of any phase transition. This questions has been addressed elsewhere in the context of phase transition in nonequilibrium, or driven, classical systems.(46) In the remainder of this paper we study the analogous question in the context of quantum, or zero-temperature, phase transitions in equilibrium systems.

3. GENERIC SCALE INVARIANCE AND PHASE TRANSITIONS IN QUANTUM SYSTEMS

3.1. Field Theory

We now turn to a quantum fluid, specifically, to systems of interacting electrons in solids. Since we will again be interested in effects at long wavelengths and long times, the ionic lattice and effects resulting from it will be irrelevant for our purposes, and we therefore adopt a simple "jellium" model of free electrons interacting via the Coulomb interaction in the presence of a charge neutralizing homogeneous background. There are various ways to theoretically deal with this many-body problem, but the most powerful one for our purposes is a field-theoretic formulation of quantum statistical mechanics. This method(19) starts by writing the partition function in terms of a functional integral,

$$Z = \int_{\psi(0) = -\psi(1/k_{\mathrm{B}}T)} D[\bar{\psi}, \psi]\, e^{S[\bar{\psi}, \psi]}. \tag{35}$$

Here D is a functional integration measure with respect to fermionic, i.e., Grassmann valued, fields $\bar{\psi}$ and ψ that are defined on a real-space/imaginary-time manifold with the time sector spanning the interval $[0, 1/k_{\mathrm{B}}T]$. The action S for our model system reads

$$S = \int dx \sum_{\sigma} \bar{\psi}_{\sigma}(x)\, [-\partial_{\tau} + \nabla^2/2m + \mu]\, \psi_{\sigma}(x) + S_{\mathrm{int}}. \tag{36}$$

Here $x \equiv (\mathbf{x}, \tau)$ with $\mathbf{x}$ denoting position, τ imaginary time, and σ is the spin index. $\int dx \equiv \int d\mathbf{x} \int_0^{1/k_B T} d\tau$, m is the electron mass, and μ is the chemical potential. At $T = 0$, $\mu = k_F^2/2m$ with k_F the Fermi wavenumber. Here and in what follows we use units such that $\hbar = 1$. At the most basic level of the theory, S_{int} represents the Coulomb potential, but it often is advantageous to integrate out certain degrees of freedom to arrive at effective, short-ranged interactions.(47) For our purposes we do not need to specify the precise form of S_{int}, it will suffice to postulate that the ground state of the interacting system is a Fermi liquid.

It is useful to go to a Fourier representation with wavevectors $\mathbf{k}$ and fermionic Matsubara frequencies $\omega_n = 2\pi T(n + 1/2)$,

$$\psi_{n,\sigma}(\mathbf{k}) = \sqrt{T/V} \int dx\, e^{-i(\mathbf{kx} - \omega_n \tau)} \psi_\sigma(x), \tag{37}$$

$$\bar{\psi}_{n,\sigma}(\mathbf{k}) = \sqrt{T/V} \int dx\, e^{i(\mathbf{kx} - \omega_n \tau)} \bar{\psi}_\sigma(x). \tag{38}$$

From Eq. (36) it follows that, in a free-electron system, single-particle excitations about the Fermi surface are soft with a linear dispersion,

$$\langle \bar{\psi}_{n,\sigma}(\mathbf{k})\, \psi_{m,\sigma'}(\mathbf{p}) \rangle_{\substack{q \to 0 \\ \omega_n \to 0}} = \delta_{\mathbf{k},\mathbf{p}}\, \delta_{nm}\, \delta_{\sigma\sigma'} \frac{Z}{i\omega_n - v_F q}, \tag{39}$$

with $Z = 1$, $q = |\mathbf{k}| - k_F$, and $v_F = k_F/m$ the Fermi velocity. Here $\langle \cdots \rangle = \int D[\bar{\psi}, \psi] \ldots e^S/Z$ comprises both a quantum mechanical and a thermodynamic average. Fermi liquid theory shows that Eq. (39) remains valid in the presence of S_{int}, only the values of Z and v_F are changed compared to a free electron gas.(48) Similarly, there are soft two-particle excitations. For instance, the susceptibility

$$\sum_{\mathbf{k},\mathbf{p}} \sum_{n',m'} \langle (\bar{\psi}_{n,\sigma}(\mathbf{k}+\mathbf{q})\, \psi_{m,\sigma}(\mathbf{k}))(\bar{\psi}_{n',\sigma'}(\mathbf{p}-\mathbf{q})\, \psi_{m',\sigma'}(\mathbf{p})) \rangle \tag{40}$$

diverges for $\mathbf{q} = 0$, $nm < 0$, and $\Omega_{n-m} \equiv \omega_n - \omega_m \to 0$ like $1/|\Omega_{n-m}|$. At nonzero external wavenumbers $|\mathbf{q}|$, the wavenumber scales like the frequency, so the dispersion of the soft modes is again linear.

In many solid-state electronic systems, quenched disorder is present in the form of impurities or lattice defects. This has important consequences for the transport and thermodynamic properties of the system, and we model it by means of a term in the action

$$S_{\text{dis}} = \int dx\, u(\mathbf{x}) \sum_\sigma \bar{\psi}_\sigma(x)\, \psi_\sigma(x). \tag{41}$$

Here $u(\mathbf{x})$ is a static, random potential governed by a distribution $P[u]$. For simplicity, we take $u(\mathbf{x})$ to be delta-correlated and Gaussian distributed with the second moment of P given by

$$\{u(\mathbf{x})\, u(\mathbf{y})\}_{\text{dis}} = \frac{1}{\pi N_{\text{F}} \tau_{\text{rel}}}\, \delta(\mathbf{x}-\mathbf{y}), \tag{42}$$

Here $\{\cdots\}_{\text{dis}}$ denotes the disorder average, τ_{rel} is the elastic relaxation time, and N_{F} is the density of states at the Fermi surface. In the presence of S_{dis}, the single-particle excitations, Eq. (39), are obviously massive with a mass proportional to $1/\tau_{\text{rel}}$. However, two-particle excitations are still soft. In contrast to the clean case, their dispersion relation is diffusive. For instance, the disorder averaged analog of the two-particle propagator given in Eq. (40) has the form$^{(23)}$

$$\sum_{\mathbf{k},\,\mathbf{p}} \sum_{n',\,m'} \{\langle(\bar{\psi}_{n,\sigma}(\mathbf{k}+\mathbf{q})\, \psi_{m,\sigma}(\mathbf{k}))(\bar{\psi}_{n',\sigma'}(\mathbf{p}-\mathbf{q})\, \psi_{m',\sigma'}(\mathbf{p}))\rangle\}_{\text{dis}} \propto \frac{1}{|\Omega_{n-m}|+D\mathbf{q}^2}, \tag{43}$$

where the diffusion coefficient D is proportional to τ_{rel}. In both the clean and the disordered cases, the two-particle excitations have the same form for different combinations of the spin indices.

Notice that the softness of the two-particle excitations is not related to any conservation law, i.e., the combinations of fermion fields do not correspond to a density or, in the clean case, a current. Rather, these excitations are the Goldstone modes resulting from the spontaneous breaking of the symmetry between positive and negative Matsubara frequencies. The order parameter that belongs to this symmetry is the quantity

$$Q = \lim_{\omega_n \to 0+} \langle \bar{\psi}_{n,\sigma}(\mathbf{x})\, \psi_{n,\sigma}(\mathbf{x})\rangle - \lim_{\omega_n \to 0-} \langle \bar{\psi}_{n,\sigma}(\mathbf{x})\, \psi_{n,\sigma}(\mathbf{x})\rangle, \tag{44}$$

which is the single-particle spectral function, or the difference between the retarded and advanced Green functions. Since the causal Green function has a cut on the real axis, Q is nonzero as long as the density of states at the Fermi surface is nonzero. This connection between the single-particle spectral function and the soft mode spectrum was discovered by Wegner for the disordered case,$^{(49)}$ and has been elaborated on in refs. 50, 51, and 47.

3.2. Long-Time Tails and Spatial GSI in Equilibrium

3.2.1. Long-Time Tails, a.k.a. Weak-Localization Effects

At zero temperature, a clean electron system has an infinite diffusivity or conductivity, so we start our discussion with the disordered case. One expects the mode-coupling argument given in Section 2 to still apply, with the transverse-velocity modes of Eqs. (14) replaced by the diffusive modes of Eq. (43). Since the dispersion relation is the same in both the classical fluid case and the disordered electron case, we thus expect a LTT in the real part of the conductivity that takes the same functional form as Eq. (19), viz.

$$\sigma(\omega)/\sigma_0 = 1 - c_d^\sigma\, \omega^{(d-2)/2}. \tag{45}$$

This expectation is indeed borne out by an explicit calculation for both noninteracting(52) and interacting(53) electrons. It turns out, however, that the coefficient c_d^σ is *negative*, in contrast to the classical case. This difference in the sign of the LTT is due to the fact that the scatterers that lead to a finite conductivity in the quantum case are static, while in the classical fluid they are the moving fluid particles themselves. The strength of the LTT, on the other hand, is the same in both cases due to the strong similarity of the respective soft mode spectra.

In this context it is interesting to point out that another, and in some sense closer, classical analog of the quenched disordered electron fluid is the classical Lorentz gas, which consists of noninteracting classical particles moving between static scatterers.(54) The diffusivity in this case also shows a LTT with the same sign as in the quenched disordered quantum case (and for the same reason), but the strength of the LTT is weaker, viz. $\omega^{d/2}$ in frequency space or $1/t^{(d+2)/2}$ in time space. The reason is that in the classical Lorentz gas there is no analog of the spontaneously broken symmetry mentioned above, and the only soft mode (which is diffusive) is due to particle number conservation.(55) There are thus many fewer soft modes than in either a classical fluid or a disordered electron system, which leads to a weaker mode-coupling effect and, hence, to a weaker LTT. Indeed, at nonzero temperature the $\omega^{(d-2)/2}$ dependence in Eq. (45) gets transformed into a $T^{(d-2)/2}$ behavior, and in the limit $\omega \to 0$ at fixed $T > 0$ one recovers the result for the classical Lorentz gas. This example illustrates that for the purpose of comparing LTT effects in different systems the crucial criterion is how similar the soft-mode spectra are. In this respect the disordered electron system, even without electron-electron interactions, is closer to the classical fluid than to its classical limit, the classical Lorentz gas.

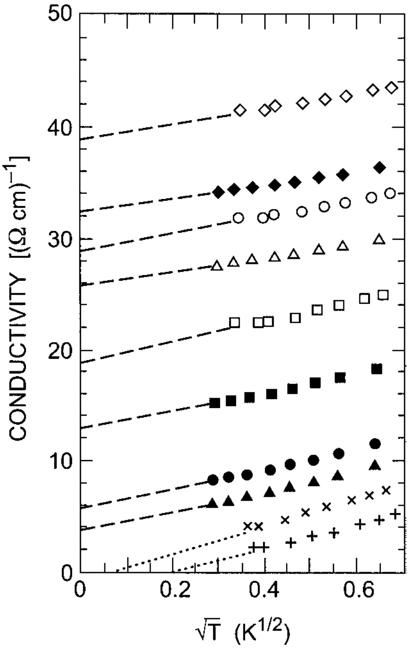

Fig. 6. Conductivity data from ref. 56 showing the static conductivity of ten Si : B samples in a magnetic field plotted against $\sqrt{T}$ at low temperatures. From ref. 23.

The fact that a nonzero temperature cuts off the LTT singularity provides the most convenient way to experimentally observe the phenomenon in electronic systems, since the dynamical conductivity is hard to measure. Fig. 6 shows the $\sqrt{T}$ dependence of $\sigma(\omega = 0, T)$ in bulk Si : B. In 2-d systems, realized by thin metallic films, the zero exponent in Eq. (45) translates into a logarithmic frequency or temperature dependence within a large dynamical range, like in the classical case, Eq. (3). An experimental example of this phenomenon is shown in Fig. 7.

These LTTs in disordered electron systems are known as "weak-localization effects." This is because they can be considered precursors of the "strong localization" that occurs in systems with very strong disorder, which undergo a phase transition from a metal to an insulator as a function of the disorder.(21) In two-dimensions the weak-localization effects are strong enough to prevent the formation of a true metal altogether,(18) and as a function of increasing disorder the system undergoes a crossover from a weak insulator to a strong one. Systems in $d \leqslant 2$ are always localized, or insulating, irrespective of the strength of the disorder, at least for noninteracting electrons.(59)

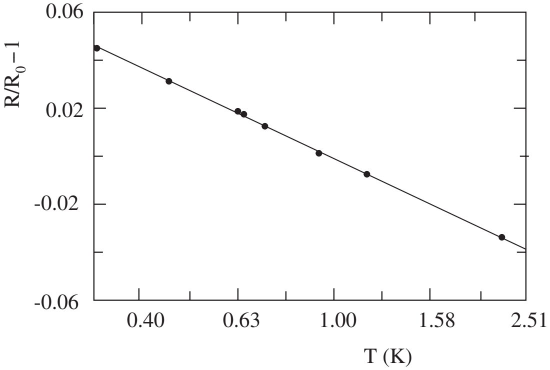

Fig. 7. Resistance data from ref. 57, for a thin PdAu film. The resistance R, normalized to $R_0 = R(T = 1\text{K})$, is plotted versus $\log T$. From ref. 58.

3.2.2. Generic Scale Invariance in Equilibrium

In quantum statistical mechanics, as opposed to the classical theory, there is an intrinsic coupling between statics and dynamics. This is manifested by the basic fermionic fields $\bar{\psi}$ and ψ being functions of both space and (imaginary) time. Consequently, one expects effects similar to the LTTs in time correlation functions to appear in static correlation functions, even in equilibrium. As an example, let us consider the wavenumber dependent static spin susceptibility χ_s in a disordered electron system. Given that the wavenumber scales like the square root of the frequency, Eq. (43), it is natural to guess that χ_s at small wavenumbers has the form

$$\chi_s(\mathbf{q}) = c_0 - c_{d-2}\,|\mathbf{q}|^{d-2} + O(\mathbf{q}^2). \tag{46}$$

The coefficients c_0 and c_{d-2} are both expected to be positive. This is because the quenched disorder slows the electrons down, which leads to an increased effective electron-electron interaction. Consequently, the disorder enhances c_0 compared to the value of the homogeneous spin susceptibility in a clean system, and the susceptibility decreases with increasing wavenumber. This is the same effect that leads to a positive coefficient c_d^{σ} in Eq. (45). This result is confirmed by explicit calculations.(60) This nonanalytic wavenumber dependence, which for dimensions $2 < d < 4$ is the dominant one, corresponds to a $1/r^{2(d-1)}$ decay in real space. We see that the coupling between statics and dynamics produce GSI in quantum systems even in equilibrium. Note that this effect is much weaker than the GSI seen in non-equilibrium classical (see Section 2.4) or quantum (see Section 3.4 below) systems.

Detailed calculations show that the origin of the GSI can be traced, as the above argument suggests, to the same Goldstone modes that lead to the LTTs and were discusssed in the previous subsection. The relevant contribution to χ_s can be schematically represented by the integral$^{(60)}$

$$\int_{|\mathbf{q}|}^{\Lambda} dp\, p^{d-1} \int d\omega \, \frac{\omega}{(\omega+T+p^2)^3}, \tag{47}$$

with Λ a microscopic wavenumber. Equation (47) demonstrates the coupling between statics and dynamics mentioned above. Notice that for power-counting purposes the integrand is again a product of two diffusive modes. At $T=0$, the integral yields Eq. (46), and at $T>0$ the temperature cuts off the leading singularity as discussed in Section 3.2.1 above. In contrast to the LTT of the previous subsection, the above results hold for interacting electron systems only. For noninteracting electrons, the absence of frequency mixing prevents the coupling of the static spin density fluctuations to the soft modes, and χ_s is analytic at zero wavenumber.

One might expect all static correlation functions to display such nonanalytic wavenumber dependences, but this turns out not to be true. For instance, the particle number density susceptibility does not have a leading singularity analogous to that in χ_s, and neither does any other spin-singlet particle-hole susceptibility like, e.g., the number density current susceptibility. This is because these observables couple less strongly to the soft modes than the spin density. In ref. 61 general criteria have been developed that allow to determine which susceptibilities show GSI due to soft modes, and which do not. This reference also discusses limitations of the equivalency between statics and dynamics in quantum statistical mechanics. For instance, it is remarkable that the conductivity in a disordered electron systems shows an $\omega^{(d-2)/2}$ LTT, see Eq. (45), but the corresponding static current susceptibility does not show an analogous $|\mathbf{q}|^{d-2}$ wavenumber dependence. This is because a finite frequency breaks the symmetry in frequency space that we mentioned in Section 3.1. As a result, dynamical current fluctuations couple more strongly to the diffusive soft modes than static ones.

We now turn to the clean case. As we have seen, the only difference compared to the disordered case is the dispersion relation of the soft modes, which now is ballistic rather than diffusive, i.e., wavenumbers scale like frequencies, which leads to weaker LTT and GSI effects. Let us again consider the static spin susceptibility. From Eq. (47) we expect that, at zero temperature, there is a contribution to χ_s that is of the form

$$\int_{|\mathbf{q}|}^{\Lambda} dp\, p^{d-1} \int d\omega \, \frac{\omega}{(\omega+p)^3}. \tag{48}$$

which leads to

$$\chi_s(\mathbf{q}) = c_0' + c_{d-1}\,|\mathbf{q}|^{d-1} + O(\mathbf{q}^2), \tag{49}$$

with positive coefficients c_0' and c_{d-1}. Explicit calculations confirm this result.(62, 63) As in the case of the LTTs, the signs of the nonanalyticities in the clean and disordered cases, respectively, are different. A simple physical explanation in the present case is that the nonanalyticity is produced by fluctuation effects that weaken the tendency towards ferromagnetism. Consequently, they decrease the value of $\chi_s(\mathbf{q}=0)$, and with increasing wavenumber the susceptibility increases.

3.3. Quantum Phase Transitions

In Section 2.3 we have seen how LTTs affect the critical behavior at a classical phase transition. Here we discuss the analogous problem for a quantum phase transition.(24, 25, 26) An example is the ferromagnetic transition that is observed in, e.g., MnSi,(64) UGe_2,(65) or $ZrZn_2$(66) at $T=0$ as a function of hydrostatic pressure. A crucial difference between the quantum and classical cases is that in the latter, due to the lack of coupling between statics and dynamics, the LTTs affect only the critical dynamics and the critical behavior of quantities given by time correlation functions, like the thermal conductivity in the example in Section 2.3. In the quantum case, however, the LTT/GSI phenomena also influence the critical behavior of thermodynamic quantities. The easiest way to see this is by deriving an order-parameter or Landau–Ginzburg–Wilson (LGW) theory for the phase transition.(67)

3.3.1. Order-Parameter Field Theory

Let us consider a phase transition with an order-parameter field $n(x) = n(\bar{\psi}(x), \psi(x))$ that is bilinear in the fermion fields. For example, in the ferromagnetic case, n is a vector field, viz. the spin density $n_i(x) = \sum_{\sigma,\sigma'} \bar{\psi}_\sigma(x)(\sigma_i)_{\sigma,\sigma'}\psi_{\sigma'}(x)$, with σ_i $(i = x, y, z)$ the Pauli matrices. In general, n is a rank-m tensor field. In order for a phase transition to a phase with a nonvanishing expectation value of n to occur, the interaction term S_{int} in the action must contain an interaction between the order-parameter modes. This term, which we denote by $S_{\text{int}}^{\text{OP}}$, reads schematically

$$S_{\text{int}}^{\text{OP}} = \Gamma \int dx\, n^2(x), \tag{50}$$

with Γ a coupling constant. For simplicity, here and in what follows we suppress both the spin and tensor labels, and use an obvious symbolic notation. The full action we write as

$$S = S_0 + S_{\text{int}}^{\text{OP}}, \tag{51}$$

with the action S_0, which we will refer to as the "reference ensemble," containing all contributions other than $S_{\text{int}}^{\text{OP}}$.

We now follow Hertz^(68)^ in deriving an order-parameter field theory for the phase transition under consideration. To this end, we first decouple $S_{\text{int}}^{\text{OP}}$ by means of a Hubbard–Stratonovich transformation.(69) Denoting the Hubbard–Stratonovich field by M, we write the partition function

$$\begin{aligned} Z &= \int D[\bar{\psi}, \psi]\, e^{S[\bar{\psi}, \psi]} \\ &= \text{const.} \times \int D[M]\, e^{-\Gamma \int dx\, M^2(x)} \langle e^{-2\Gamma \int dx M(x)\, n(x)} \rangle_0, \\ &\equiv \text{const.} \times \int D[M]\, e^{-\Phi[M]}, \end{aligned} \tag{52}$$

where $\langle \cdots \rangle_0$ denotes an average with the reference ensemble action S_0, and $\Phi[M]$ is the LGW functional. The latter reads explicitly

$$\Phi[M] = \Gamma \int dx\, M^2(x) - \ln \langle e^{-2\Gamma \int dx\, M(x)\, n(x)} \rangle_0, \tag{53}$$

and can be expanded in powers of M,

$$\begin{aligned} \Phi[M] = \frac{1}{2} \int dx_1\, dx_2\, M(x_1) &\left[\frac{1}{\Gamma}\, \delta(x_1 - x_2) - \chi^{(2)}(x_1 - x_2) \right] M(x_2) \\ &+ \frac{1}{3\,!} \int dx_1\, dx_2\, dx_3\, \chi^{(3)}(x_1, x_2, x_3)\, M(x_1)\, M(x_2)\, M(x_3) \\ &+ O(M^4), \end{aligned} \tag{54}$$

where we have scaled M with $1/\sqrt{2}\,\Gamma$. The coefficients $\chi^{(l)}$ in the Landau expansion, Eq. (54), are connected l-point correlation functions of $n(x)$ in the reference ensemble,

$$\chi^{(l)}(x_1, \ldots, x_l) = \langle n(x_1) \cdots n(x_l) \rangle_0^c. \tag{55}$$

3.3.2. Local Versus Nonlocal LGW Theories

A crucial question now arises concerning the behavior of the correlation functions defined in Eq. (55). Suppose their Fourier transforms are finite in the limit of small wavenumbers and frequencies. In lowest order in a gradient expansion one can then simply localize the cubic and higher-order coefficients. Defining a Fourier transform of the Hubbard–Stratonovich field by

$$M(q) = \sqrt{T/V} \int dx \, e^{-i\mathbf{q}\mathbf{x} + i\omega_n \tau} M(x). \tag{56}$$

we obtain an LGW functional

$$\begin{aligned} \Phi[M] &= \frac{1}{2} \sum_q M(q) \, [1/\Gamma - \chi^{(2)}(q)] \, M(-q) \\ &\quad + \frac{u_4}{4!} \frac{T}{V} \sum_{q_1, \ldots, q_4} \delta(q_1 + q_2 + q_3 + q_4) \, M(q_1) \, M(q_2) \, M(q_3) \, M(q_4) \\ &\quad + \cdots \end{aligned} \tag{57}$$

Here the four-vector $q = (\mathbf{q}, \omega_n)$ comprises wavevector and Matsubara frequency, and we have assumed that $\chi^{(3)}$ vanishes at zero frequency and zero wavenumber. Suppose $\chi^{(2)}$ is an analytic function of the wavenumber,

$$\chi^{(2)}(q) = c_0 + c_2 \mathbf{q}^2 + c_\Omega \, |\Omega_n| / |\mathbf{q}|^m + \cdots, \tag{58}$$

where the value of m depends on the specific system. Then we obtain an ordinary Landau theory with the Gaussian part of the LGW functional given by

$$\Phi^{(2)}[M] = \tfrac{1}{2} \sum_q M(q) \, [r + a_2 \, \mathbf{q}^2 + a_\Omega \, |\Omega_n| / |\mathbf{q}|^m] \, M(-q). \tag{59}$$

Here $r = 1/\Gamma - \chi^{(2)}(q = 0)$ is the bare distance from the critical point at $T = 0$. Equation (59) leads to mean-field values for the critical exponents ν and γ, and a dynamical critical exponent $z = m + 2$. Power counting shows that the non-Gaussian terms are irrelevant in the renormalization-group sense of the word and, hence, the exact critical behavior is also mean field-like. This is the conclusion that was reached by Hertz,(68) namely, that generically quantum phase transitions have mean-field critical behavior.

As we have seen in Section 3.2.2, however, this scenario does not necessarily apply. Equation (46) provides an example of a susceptibility that does not have the form of Eq. (58) due to GSI effects. Moreover, if the nonanalytic wavenumber dependence of the susceptibility $\chi^{(2)}$ is caused by soft modes that are made massive by an external field H conjugate to the order parameter (or by a nonvanishing average of the order parameter), then the singular part of the static, field-dependent susceptibility will have the form

$$\chi^{(2)}_{\text{sing}}(\mathbf{q}, \Omega_n = 0, H) = (|\mathbf{q}|^x + H)^y, \tag{60}$$

where $y > 0$ and $x > 0$ are exponents that determine the nature of the nonanalyticity and the scaling of H with the wavenumber, respectively. This is true, for instance, in the case of the disordered itinerant ferromagnet, where $x = 2$ and $y = (d-2)/2$. Since the higher susceptibilities $\chi^{(3)}$, $\chi^{(4)}$, etc., can be obtained from $\chi^{(2)}$ by differentiating with respect to H, this implies that they diverge in the limit of zero wavenumbers and frequencies. That is, the gradient expansion that leads to the usual LGW theory, and hence LGW theory itself, do not exist. Instead, one obtains a field theory where the coefficients of the various powers of the field M are singular functions of wavenumber and frequency. In other words, the field theory is not local.(60) The above arguments make it clear that this is a direct consequence of the GSI in the equilibrium quantum system.

Such nonlocal field theories are difficult to handle and not suitable for explicit calculations. For the purpose of determining the critical behavior one can try to circumvent this problem by checking whether the field theory still allows for a Gaussian critical fixed point with respect to which the higher-order terms are irrelevant by power counting. This was the approach taken in ref. 60 for the disordered ferromagnetic transition, and it relied strictly on power counting by taking the divergence of, e.g., the coefficient u_4 in Eq. (57) into account by assigning it a suitable scale dimension. It turns out that this procedure yields the correct power laws, but misses logarithmic corrections to scaling that power counting is not sensitive to. In the following subsection we use this example to illustrate the interplay between GSI and quantum critical behavior.

3.3.3. *Example 1: The Disordered Quantum Ferromagnetic Transition*

As an example of the interplay between LTT/GSI effects and quantum critical phenomena, we now consider the quantum critical behavior at the ferromagnetic transition in quenched disordered itinerant electron systems in d-dimensions. As we mentioned in the last subsection, in that case the two-point susceptibility $\chi^{(2)}$ has the form

$$\chi^{(2)}(q) = c_0 + c_{d-2}\,|\mathbf{q}|^{d-2} + c_2\mathbf{q}^2 + c_\Omega\,|\Omega_n|/\mathbf{q}^2 + \cdots. \tag{61}$$

This leads to a Gaussian part of the LGW functional

$$\Phi^{(2)}[M] = \frac{1}{2}\sum_{q} M(q)\,[r + a_{d-2}\,|\mathbf{q}|^{d-2} + a_2\,\mathbf{q}^2 + a_{\Omega}\,|\Omega_n|/|\mathbf{q}|^2]\,M(-q). \qquad (62)$$

Furthermore, the field-dependent susceptibility has the form given by Eq. (60) with $x=2$ and $y=(d-2)/2$. This implies that u_4 in Eq. (57), and all of the coefficients of higher-order terms in the Landau expansion, do not exist in the limit of small wavenumbers and frequencies. For instance, u_4 behaves in this limit like

$$u_4(|\mathbf{q}| \to 0, \Omega_n = 0) = u_4^{(d-6)}/|\mathbf{q}|^{d-6} + u_4^{(0)}, \qquad (63)$$

with $u_4^{(d-6)}$ and $u_4^{(0)}$ finite numbers.

Let us now look for a Gaussian fixed point that describes the critical behavior. From Eq. (62) one reads off the Gaussian values of the correlation length exponent ν, the susceptibility exponents γ and η, and the dynamical exponent z as

$$\nu = 1/(d-2), \qquad \gamma = 1, \qquad \eta = 4-d, \qquad z = d, \qquad (64)$$

for $2 < d < 4$, and

$$\nu = 1/2, \qquad \gamma = 1, \qquad \eta = 0, \qquad z = 4, \qquad (65)$$

for $d > 4$. In $d = 4$ one finds mean-field exponents with logarithmic corrections to scaling. Power counting suggests that the non-Gaussian terms in the action are irrelevant with respect to this fixed point, so that these exponents represent the exact critical behavior. However, for the remaining exponents, the order-parameter exponents β and δ, one needs to take into account these terms, as is the case in ordinary Landau theory. From Eq. (60) with $x = 2$ we see that the average order parameter m, which scales like H, scales like the wavenumber squared. The coefficient u_4, Eq. (63), thus scales like $m^{-d/2}$, which suggests an equation of state

$$r\,m + v\,m^{d/2} + u\,m^3 = H, \qquad (66)$$

with $u, v > 0$. From this we obtain

$$\beta = 2/(d-2), \qquad \delta = d/2, \qquad (67)$$

for $2 < d < 6$, and

$$\beta = 1/2, \qquad \delta = 3, \qquad (68)$$

for $d > 6$. In $d = 6$ there are again logarithmic corrections to scaling. A more sophisticated analysis$^{(60)}$ confirms these results, which illustrate the strong influence of the GSI on the critical behavior: The exponents given in Eqs. (64) and (67) for the experimentally most interesting case $d = 3$ are very different from the mean-field exponents one obtains if one neglects the GSI effects.$^{(68)}$

While the above simple power-counting arguments produce the correct exponents of the exact critical behavior, it turns out that the simple Gaussian fixed point presented above is marginally unstable. This has been traced to the existence of two time scales, viz. the critical one with dynamical exponent z, and the diffusive one with dynamical exponent 2. The existence of the latter has been obscured in the derivation of the LGW theory, which integrates out the diffusive modes. A more careful analysis, which keeps all of the soft modes explicitly and on equal footing, shows that the actual critical fixed point is not Gaussian, but has the same exponents as given above, with multiplicative logarithmic corrections to scaling.$^{(70)}$ A convenient way to account for these corrections is to write the critical behavior as power laws, with scale-dependent critical exponents. For instance, the correlation length ξ as a function of the distance r from criticality has a log-log normal correction to the simple power law,

$$\xi \propto 1/r^{1/(d-2)}\, g(\ln(1/r)), \tag{69}$$

where $g(x)$ is a function whose leading behavior at large arguments is

$$g(x \to \infty) = \text{const.} \times e^{[\ln(c(d)\, x)]^2/2\ln(d/2)}, \tag{70}$$

with $c(d)$ a dimensionality dependent constant. This behavior can be represented by writing $\xi \propto r^{-1/\nu}$ with a scale dependent exponent ν,

$$1/\nu = d - 2 + \ln g(b)/\ln b, \tag{71}$$

with b an arbitrary renormalization group length scale factor. For instance, to translate the dependence of b into an r-dependence, one needs to substitute $b = r^{-\nu}$. For the other exponents, the analysis of ref. 70 yields

$$z = 2\delta = d + \ln g(b)/\ln b, \tag{72}$$

$$\eta = 4 - d - \ln g(b)/\ln b, \tag{73}$$

$$\beta = 2\nu. \tag{74}$$

Equations (68) and (70) are valid for $2 < d < 4$. The exponent γ has no logarithic corrections, and is $\gamma = 1$ exactly, as stated in Eqs. (63).

The above detailed predictions have yet to be tested experimentally.

3.3.4. Example 2: The Clean Quantum Ferromagnetic Transition

As a second example we briefly consider the case of a clean ferromagnet. In this case, the coefficient u_4 of the quartic term in the Landau expansion diverges like $1/|\mathbf{q}|^{d-3}$, and the average order parameter m scales like the wavenumber. Furthermore, as we have mentioned in Section 3.2.2, the sign of the nonanalytic term is opposite to that in the disordered case. This leads to an equation of state

$$r\,m - v\,m^d + u\,m^3 = H, \tag{75}$$

for general d, and

$$r\,m + v\,m^3 \ln m^2 + u\,m^3 = H, \tag{76}$$

for $d = 3$, with $u, v > 0.$(71)

These considerations suggest that, generically, the quantum ferromagnetic transition of clean itinerant electrons is of first order. Indeed, this is what is observed in MnSi.(64) see Fig. 8, and in UGe_2,(65) both systems where the ferromagnetic transition can be triggered at very low temperatures by applying hydrostatic pressure. Both a nonzero temperature and nonzero disorder cut off the singularity in Eq. (76), and disorder, of course, induces the stronger singularity shown in Eq. (66). For small values of both

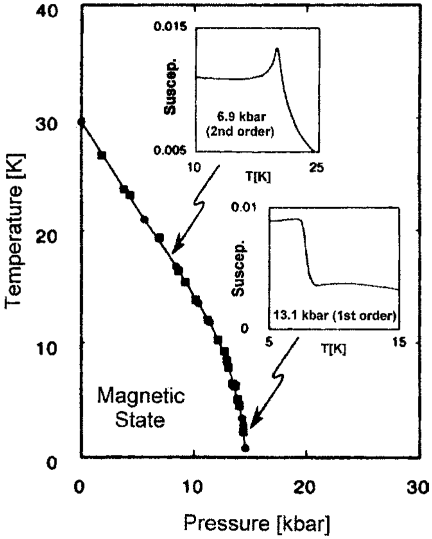

Fig. 8. Phase diagram of MnSi. The insets show the behavior of the susceptibility close to the transition as reported in ref. 64. From ref. 72.

temperature and disorder, the competition between these terms leads to an interesting structure of the phase diagram, with tricritical points and critical endpoints. This has been discussed in detail in ref. 71.

3.4. Nonequilibrium Effects

Very recently, spatial correlations of density fluctuations have been studied in noninteracting disordered electronic system that are not in equilibrium.(73) For the model defined by Eqs. (36) and (40), without the electron-electron interaction term S_{int} but in the presence of a chemical-potential gradient $\nabla\mu$, this reference calculated an electron density-scatterer density correlation function defined by

$$C_1(\mathbf{x}, \mathbf{y}) = \{\langle \delta n(\mathbf{x}) \rangle\, u(\mathbf{y})\}_{\text{dis}}, \tag{77}$$

where $\langle \cdots \rangle$ denotes a nonequilibrium thermal average. The calculation shows that the Fourier transform of Eq. (77), $C_1(\mathbf{k})$, behaves like

$$C_1(\mathbf{k}) = \frac{i\mathbf{k}\cdot\nabla\mu}{2\pi D\mathbf{k}^2}. \tag{78}$$

In real space, this corresponds to a decay proportional to $1/|\mathbf{x}|^{d-1}$. This is the same result as the one in the corresponding classical Lorentz gas, since the additional soft modes in the quantum system do not couple to C_1.

It is also interesting to consider the electronic structure factor,

$$C_2(\mathbf{x}, \mathbf{y}) = \{\langle \delta n(\mathbf{x})\, \delta n(\mathbf{y}) \rangle\}_{\text{dis}}. \tag{79}$$

Let us consider the nonequilibrium part of C_2. As one would expect, the calculation yields a result that is analogous to the one in the classical fluid, Eq. (29), viz.

$$C_2(\mathbf{k}) = \frac{N_{\text{F}}\mu\tau_{\text{rel}}}{6d\pi(D\mathbf{k}^2)^2}\,[25(\nabla\mu)^2 - 12(\hat{\mathbf{k}}\cdot\nabla\mu)^2]. \tag{80}$$

As in the classical case, this corresponds to a decay in real space like const. $-|\mathbf{x}|$ in three-dimensions. For a classical Lorentz gas one finds instead

$$C_2(\mathbf{k}) \propto (\nabla\mu)^2/\mathbf{k}^2. \tag{81}$$

The weaker singularity in this case is due to the fact that the classical model has fewer soft modes, as we discussed in Section 3.2.1.

4. CONCLUSIONS

In this paper we have emphasized that the soft modes which always exist in many-body systems due to either conservation laws or broken continuous symmetries via Goldstone's theorem, generically couple to the correlation functions relevant for both scattering and transport experiments, and lead to power-law correlations in space and time, in the entire phase diagram. That is, correlation functions usually exhibit generic scale invariance.

In the classical case we have seen that these GSI effects can get amplified either in a phase with Goldstone modes, or near continuous phase transitions. In both cases the crucial point is that the coefficient of a correlation function exhibiting GSI is proportional to a static susceptibility that diverges either in an entire Goldstone phase, or at a special critical point.

The quantum case is of particular interest because of the inherent coupling between statics and dynamics in zero-temperature systems. Generically, this implies that systems that exhibit GSI in the time domain do so in space as well. This has been contrasted to classical systems, where the equilibrium fluid provides an example of a system that exhibits GSI in time correlations functions, i.e., that has LTTs, but exponentially decaying correlations in space. This coupling also leads to the important conclusion that many quantum phase transitions are in a non-mean field universality class governed by long-ranged interactions.

Among the many experimental consequences of the particular type of GSI discussed here we mention, (1) transport coefficients that depend nonanalytically on the frequency, in both classical and quantum systems, (2) singularities in transport coefficients near continuous phase transitions, (3) enhanced, compared to equilibrium, light scattering in nonequilibrium systems, and, (4) quantum phase transitions that are in different universality classes than they would be in the absence of these effects. The first three of these effects have been observed. Regarding the fourth one, the predicted first order nature of the quantum phase transition in clean systems has been observed, but the critical behavior in disordered systems remains to be investigated experimentally.

ACKNOWLEDGMENTS

It is with great pleasure that we dedicate this paper to Professor J. Robert Dorfman on the occasion of his sixty-fifth birthday.

This work was supported by National Science Foundation Grants Nos. DMR-98-70597 and DMR-99-75259. The research of J. V. Sengers is supported by the Chemical Sciences, Geosciences and Biosciences Division of the Office of Basic Energy Sciences of the Department of Energy under Grant No. DE-FG02-95ER-14509.

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